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GIBBS-MARKOV-YOUNG STRUCTURES WITH (STRETCHED) EXPONENTIAL TAIL FOR PARTIALLY HYPERBOLIC ATTRACTORS

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Abstract. We study a partially hyperbolic set K on a Riemannian manifold M whose tangent space splits as $T_K M = E^{cu} \oplus E^s$, for which the center-unstable direction E^{cu} is non-uniformly expanding on some local unstable disk. We prove that the (stretched) exponential decay of recurrence times for an induced scheme can be deduced under the assumption of (stretched) exponential decay of the time that typical points need to achieve some uniform expanding in the center-unstable direction. This extends a result in [7] to the (stretched) exponential case. As an application of our main result we obtain (stretched) exponential decay of correlations and exponentially large deviations for a class of partially hyperbolic diffeomorphisms introduced in [1].

Contents

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1. Introduction

In the late 60's and 70's, Sinai, Ruelle and Bowen brought Markov partitions and symbolic dynamics into the theory of uniformly hyperbolic systems; see [18, 8, 17]. Ruelle wrote: 'This allowed the powerful techniques and results of statistical mechanics to be applied into smooth dynamics' in [9, Preface]. To study the systems beyond uniformly hyperbolic, Young used Markov partition to build Young tower in [20, 21] for systems with nonuniform hyperbolicity, including Axiom A attractors, piecewise hyperbolic maps, billiards with convex scatterers, logistic maps, intermittent maps and Hénon-type attractors. Under these towers, Young studied some statistical properties of the non uniformly hyperbole systems, including the existence of SRB measures, exponential decay of correlation and the validity of the Central Limit Theorem for the SRB measure. Roughly speaking, a Markov structure is characterized by some selected region of the phase space that is divided into an at most countable number of subsets with associated *recurrence times*. Young called it 'horseshoe with infinitely many branches'. These structures have some properties which address to Gibbs states and for that reason they are nowadays sometimes referred to as Gibbs-Markov-Young (GMY) structures; see Definition 1.5.

In [10], Bonatti and Viana considered partially hyperbolic attractors with mostly contracting direction, i.e. the tangent bundle splitting as $E^{cs} \oplus E^u$, with the E^u direction uniformly expanding and the E^{cs} direction mostly contracting (negative Lyapunov exponents). They proved the existence of an SRB measure under those conditions. In [12], Castro showed the existence of GMY structure, thus obtaining statistical properties like exponential decay of correlations and the validity of the Central Limit Theorem. The Central Limit Theorem for these systems has also been obtained by Dolgopyat in [13].

However, as most of the richness of the dynamics in partially hyperbolic attractors appears in the unstable direction, the case $E^{cu} \oplus E^s$ (now with the stable direction being uniform and the unstable nonuniform) comprises more difficulties than the case $E^{cs} \oplus E^u$. The existence of SRB for some classes of partially hyperbolic attractors of the type $E^{cu} \oplus E^s$ has been proven by Alves, Bonatti and Viana in [1]. In [7], Alves and Pinherio obtained a GMY structure quite similar to that in [5] for non-uniformly expanding (NUE) systems. Given the lack of expansion of the system at time n is polynomial small, they got polynomial decay of recurrence times and polynomial decay of correlations. Their approach, originated from [20] for Axiom A attractors, has shown to be not efficient enough to estimate the tail of recurrence times for non-uniformly hyperbolic systems with exponential tail of hyperbolic times. This is due to the fact that at each step of their algorithmic construction just a definite fraction of hyperbolic times is used.

In [14], Gouëzel developed a new construction with more efficient estimate for return times. As a starting point, Gouëzel used the fact that the attractor could be partitioned into finite number of sets with small size. That gave rise to more precise calculations yielding also the (streched) exponential case. However, it is not clear that Gouëzel strategy has a direct application to the partially hyperbolic setting $E^{cu} \oplus E^s$, because the attractor is typically made of unstable leaves, which are not bounded in their intrinsic distance. Partially inspired by [14, 16], Alves, Dias and Luzzatto gave in [2] an improved *local* GMY structure, with much more efficiency than [5] in the use of hyperbolic times that made it possible to prove the integrability of recreance times under very general conditions.

The aim of this work is to fill a gap in the theory of partially hyperbolic diffeomorphisms with centre unstable direction, where GMY structures are only known with polynomial tail of recurrence times, after [7]. From that we get (stretched) exponential decay of correlation and exponential large deviations, by the related results in [20, 6, 15]. Our strategy is based in a mixture of arguments from [2] and [14]. We construct a GMY structure by a method similar to [2] and we apply the estimates process in [14] to our GMY structure. To improve the efficiency of the algorithm in [2], our method has a main difference, namely, we keep track of all points with hyperbolic time at a given iterate and not just of a proportion of those points.

1.1. **Gibbs-Markov-Young structures.** Here we recall the structures which have been introduced in [20]. Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M, Leb (Lebesgue measure) the normalized Riemannian volume on the Borel sets of M. Given a submanifold $\gamma \subset M$, and Leb_γ denotes the Lebesgue measure on γ induced by the restriction of the Riemannian structure to γ.

Definition 1.1. An embedded disk $\gamma \subset M$ is called an *unstable manifold* if for all $x, y \in \gamma$

$$\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \to 0 \quad \text{as } n \to \infty.$$

Similarly, γ is called a *stable manifold* if for all $x, y \in \gamma$

$$\mathrm{dist}(f^{n}(x), f^{n}(y)) \to 0, \quad \text{as } n \to \infty.$$

Definition 1.2. Given $n \geq 1$, let D^u be a unit disk in $\mathbb{R}^n$ and let $\mathrm{Emb}^1(D^u, M)$ be the space of C^1 embeddings from D^u into M. A *continuous family of C^1 unstable manifolds* is a set Γ^u of unstable disks γ^u satisfying the following properties: there is a compact set K^s and a map $\Phi^u\colon K^s \times D^u \to M$ such that

(1) $\gamma^u = \Phi^u(\{x\} \times D^u)$ is a local unstable manifold;
(2) Φ^u maps $K^s \times D^u$ homeomorphically onto its image;
(3) $x \mapsto \Phi^u|(\{x\} \times D^u)$ is a continuous map from K^s to $\mathrm{Emb}^1(D^u, M)$.

Continuous families of C^1 stable manifolds are defined analogously.

Definition 1.3. A subset $\Lambda \subset M$ has a *product structure* if, for some $n \geq 1$, there exist a continuous family of n-dimensional unstable manifolds $\Gamma^u = \cup\gamma^u$ and a continuous family of $(\dim(M) - n)$-dimensional stable manifolds $\Gamma^s = \cup\gamma^s$ such that

(1) $\Lambda = \Gamma^u \cap \Gamma^s$;
(2) each γ^s meets each γ^u in exactly one point, with the angle of γ^s and γ^u uniformly bounded away from zero.

Definition 1.4. Let $\Lambda \subset M$ have a product structure defined by families Γ^s and Γ^u. A subset $\Lambda_0 \subset \Lambda$ is an *s-subset* if Λ_0 has a hyperbolic product structure defined by families $\Gamma_0^s \subset \Gamma^s$ and $\Gamma_0^u = \Gamma^u$; *u-subsets* are defined similarly.

For $* = u, s$, given $x \in \Lambda$, let $\gamma^*(x)$ denote the element of Γ^* containing x, and let f^* denote the restriction of the map f to γ^*-disks and $|\det Df^*|$ denote the Jacobian of Df^*.

Definition 1.5. A set Λ with a product structure for which properties $(\mathbf{P_0})$-$(\mathbf{P_4})$ below hold will be called a *Gibbs-Markov-Young (GMY) structure*. From here on we assume that $C > 0$ and $0 < \beta < 1$ are constants depending only on f and Λ.

($\mathbf{P_0}$) *Lebesgue detectable*: for every $\gamma \in \Gamma^u$, we have $\mathrm{Leb}_\gamma(\Lambda \cap \gamma) > 0$;

($\mathbf{P_1}$) *Markov partition and recurrence times*: there are finitely or countably many pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots \subset \Lambda$ such that

(a) for each $\gamma \in \Gamma^u$, $\mathrm{Leb}_\gamma\big((\Lambda \setminus \cup\Lambda_i) \cap \gamma\big) = 0$;

(b) for each $i \in \mathbb{N}$ there is integer $R_i \in \mathbb{N}$ such that $f^{R_i}(\Lambda_i)$ is u-subset, and for all $x \in \Lambda_i$

$$f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}(x)) \quad \text{and} \quad f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}(x)).$$

We define the *recurrence time* function $R\colon \cup_i \Lambda_i \to \mathbb{N}$ as $R|_{\Lambda_i} = R_i$. We call $f^{R_i} : \Lambda_i \to \Lambda$ the *induced map.*

($\mathbf{P_2}$) *Uniform contraction on* Γ^s: for all $x \in \Lambda$, each $y \in \gamma^s(x)$ and $n \geq 1$

$$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^n.$$

($\mathbf{P_3}$) *Backward contraction and bounded distortion on* Γ^u: for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, and $0 \leq n < R_i$

(a) $$\mathrm{dist}(f^n(y), f^n(x)) \leq C\beta^{R_i - n}\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y));$$

(b) $$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \leq C\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y))^\eta.$$

($\mathbf{P_4}$) *Regularity of foliations*:

(a) *Convergence of* $D(f^i|\gamma^u)$: for all $y \in \gamma^s(x)$ and $n \geq 0$

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C\beta^n;$$

(b) *Absolutely continuity of* Γ^s: given $\gamma, \gamma' \in \Gamma^u$, define the holonomy map $\phi\colon \gamma \cap \Lambda \to \gamma' \cap \Lambda$ as $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous with

$$\frac{d(\phi_* \mathrm{Leb}_\gamma)}{d\,\mathrm{Leb}_{\gamma'}}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

(The notion of absolute continuity is precisely given in Section 3.7.) Under these conditions we say that $F = f^R : \Lambda \to \Lambda$ is an *induced GMY map.*

1.2. **Partially hyperbolic attractors.** Here we recall the definition of partially hyperbolic attractors with mostly expanding center-unstable direction and then we state the main theorem, Theorem A. This extends the result in [7, Theorem A] to the (stretched) exponential case.

Let $f : M \to M$ be a C^{1+} diffeomorphism of a finite dimensional Riemannian manifold M. We say that f is C^{1+} if f is C^1 and Df is Hölder continuous. A set $K \subset M$ is said to be invariant if $f(K) = K$.

Definition 1.6. A compact invariant subset $K \subset M$ has a *dominated splitting*, if there exists a continuous Df-invariant splitting $T_K M = E^{cs} \oplus E^{cu}$ and $0 < \lambda < 1$ such that (for some choice of Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \cdot \|Df^{-1} \mid E^{cu}_{f(x)}\| \leq \lambda, \quad \text{for all } x \in K. \tag{1}$$

We call E^{cs} the *center-stable bundle* and E^{cu} the *center-unstable bundle.*

Definition 1.7. A compact invariant set $K \subset M$ is called *partially hyperbolic*, if it has a dominated splitting $T_K M = E^{cs} \oplus E^{cu}$ for which E^{cs} is *uniformly contracting* or E^{cu} is

uniformly expanding, i.e. there is $0 < \lambda < 1$ such that (for some choice of a Riemannian metric on M)

$$\|Df \mid E^{cs}_x\| \leq \lambda \quad \text{or} \quad \|Df^{-1} \mid E^{cu}_{f(x)}\|^{-1} \leq \lambda, \quad \text{for all } x \in K.$$

In this work we consider partially hyperbolic sets of the same type of those considered in [1], for which the center-stable direction is uniformly contracting and the central-unstable direction is non-uniformly expanding. To emphasize that, we shall write E^s instead of E^{cs}.

Definition 1.8. Given $b > 0$, we say that f is *non-uniformly expanding* at a point $x \in K$ in the central-unstable direction, if

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{f^j(x)}\| < -b. \tag{NUE}$$

If f satisfies (NUE) at $x \in K$, then the *expansion time* function at x

$$\mathcal{E}(x) = \min \left\{ N \geq 1 \colon \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^{-1} \mid E^{cu}_{f^i(x)}\| < -b, \quad \forall\, n \geq N \right\} \tag{2}$$

is defined and finite.

$\{\mathcal{E} > n\}$ is the set of points which, up to time n, have not yet achieved exponential growth of the derivative along orbits. We call $\{\mathcal{E} > n\}$ *the tail of hyperbolic times* (at time n).

We remark that if condition (NUE) holds for every point in a subset with positive Lebesgue measure of a forward invariant set $\tilde{K} \subset M$, then $K = \cap_{n \geq 0} f^n(\tilde{K})$ contains some local unstable disk D for which condition (NUE) is satisfied Leb_D almost everywhere; see [7, Theorem A].

Theorem A. *Let $f : M \to M$ be a C^{1+} diffeomorphism with $K \subset M$ an invariant transitive partially hyperbolic set. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$ and $0 < c < 1$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then there exists $\Lambda \subset K$ with a GMY structure. Moreover, there exists $0 < d < 1$ such that $\mathrm{Leb}_\gamma\{R > n\} = \mathcal{O}(e^{-dn^\tau})$ for any $\gamma \in \Gamma^u$.*

The proof of this result will be given in Section 3.

Under the assumptions of Theorem A, the set Λ coincides with Γ^u, but there are other possibilities, e.g. Λ is a Cantor set for the Hénon attractors in [11].

In Section 4 we present an open class of diffeomorphisms for which $K = M$ is partially hyperbolic and satisfies the assumptions of Theorem A. The transitivity of the diffeomorphisms in that class was proved in [19].

1.3. **Statistical properties.** A good way of describing the dynamical behavior of chaotic dynamical systems is through invariant probability measures and, in our context, a special role is played by SRB measures.

Definition 1.9. An f-invariant probability measure μ on the Borel sets of M is called an *Sinai-Ruelle-Bowen (SRB) measure* if f has no zero Lypaunov exponents μ almost everywhere and the conditional measures of μ on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on these manifolds.

It is well known that SRB measures are *physical measures*: for a positive Lebesgue measure set of points $x \in M$,

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu, \quad \text{for any continuous } \varphi : M \to \mathbb{R}. \tag{3}$$

SRB measures for partially hyperbolic diffeomorphisms whose central direction is non-uniformly expanding were already obtained in [1]. Under the assumptions of Theorem A, we also get the existence of such measures by mean of [20, Theorem 1].

Definition 1.10. We define the *correlation functions* of observables $\varphi, \psi \colon M \to \mathbb{R}$ with respect to a measure μ as

$$\mathcal{C}_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n) d\mu - \int \varphi d\mu \int \psi d\mu \right|, \quad n \geq 0.$$

Sometimes it is possible to obtain specific rates for which $\mathcal{C}_\mu(\varphi, \psi)$ decays to 0 as n tends to infinity, at least for certain classes of observables with some regularity. See that if we take the observables as characteristic functions of Borel sets, we get the classical definition of *mixing.*

The next corollary follows from Theorem A together with [6, Theorem B]; see also [6, Remark 2.4]. Though in [6] the decay of correlations depends on some backward decay rate in the unstable direction, in our case we clearly have exponential backward contraction along that direction. So the next result is indeed an extension of [7, Corollary B] to the (stretched) exponential case.

Corollary B (Decay of Correlations)**.** *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$ and $0 < c < 1$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Then some power f^k has an SRB measure μ and there is $d > 0$ such that $\mathcal{C}_\mu(\varphi, \psi \circ f^{kn}) = \mathcal{O}(e^{-dn^\tau})$ for Hölder continuous $\varphi, \psi \colon M \to \mathbb{R}$.*

If the return times associated to the elements of the GMY structure given by Theorem A are relatively prime, i.e. $\gcd\{R_i\} = 1$, then the same conclusion holds with respect to f, i.e. for $k = 1$.

Definition 1.11. Given an observable $\varphi : M \to \mathbb{R}$, we define the *large deviation* of the time average with respect to the mean of φ as

$$\mathcal{D}_\mu(\varphi, n, \varepsilon) = \mu\left(\left\{ x \in M : \left| \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) - \int \varphi d\mu \right| > \epsilon \right\}\right).$$

Using Theorem A and [15, Theorem 4.1], we also deduce a large deviations result for the SRB measure μ of f.

Corollary C (Large Deviations)**.** *Let $f : M \to M$ be a C^{1+} diffeomorphism with an invariant transitive partially hyperbolic set $K \subset M$. Assume that there are a local unstable disk $D \subset K$ and $0 < c < 1$ such that $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn})$. Then there is $d > 0$ such that for any Hölder continuous $\varphi : M \to \mathbb{R}$ and any $\varepsilon > 0$ we have $\mathcal{D}_\mu(\varphi, n, \varepsilon) = \mathcal{O}(e^{-dn})$.*

In Corollary C we do not need to take any power of f; see the considerations in [15, Section 2.2]. It remains an interesting open question to know whether we have a similar result in the stretched exponential case; this depends only on a stretched exponential version of [15, Theorem 4.1].

Further statistical properties, as the Central Limit Theorem or an Almost Sure Invariant Principle, which have already been obtained in [7], could still be deduced form Theorem A.

2. Preliminary results

In this section we state the bounded distortion property at hyperbolic times (firstly appeared in [4]) for iterations of f over disks which are tangent to a center-unstable cone filed. The material here is mainly from [1].

Firstly we give the precise definition of center-unstable cone field. We denote the continuous extensions of E^s and E^{cu} to some neighborhood U of K by $\tilde{E}^s$ and $\tilde{E}^{cu}$. The extensions are not necessarily invariant under Df. Notice the set U will be necessary in the GMY construction; see Subsection 3.5. These extensions may not be invariant under Df.

Definition 2.1. Given $0 < a < 1$, the *center-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ *of width* a is defined by

$$C_a^{cu}(x) = \left\{v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_1\| \le a\|v_2\|\right\};$$

the *stable cone field* $C_a^s = (C_a^s(x))_{x\in U}$ *of width* a is defined similarly,

$$C_a^s(x) = \left\{v_1 + v_2 \in \tilde{E}_x^s \oplus \tilde{E}_x^{cu} \text{ such that } \|v_2\| \le a\|v_1\|\right\}.$$

We notice that the dominated splitting property still holds for the extension. Up to slightly increasing $\lambda < 1$, we fix $a > 0$ and U small enough so that the domination condition (1) still holds for any point $x \in U \cap f^{-1}(U)$ and every $v^s \in C_a^s(x)$, $v^{cu} \in C_a^{cu}(f(x))$:

$$\|Df(x)v^s\| \cdot \|Df^{-1}(f(x))v^{cu}\| \le \lambda \|v^s\|\,\|v^{cu}\|.$$

The center-unstable cone field is forward invariant

$$Df(x)C_a^{cu}(x) \subset C_a^{cu}(f(x)), \quad \text{any } x \in K,$$

and this holds for any $x \in U \cap f^{-1}(U)$ by continuity.

The cu-direction tangent bundle of the iterates of a C^2 submanifold are Hölder continuous as long as they do not leave U, with uniform Hölder constants. We only need the existence of a dominated splitting $E^{cs} \oplus E^{cu}$.

Definition 2.2. An embedded C^1 submanifold $L \subset U$ is *tangent* to the centre-unstable cone field, if $T_xL \subset C_a^{cu}(x)$, at every point $x \in L$.

Given L satisfies Definition 2.2, then $f(L)$ is also tangent to the centre-unstable cone field by the domination property so far as $f(L)$ is in U,

The tangent bundle TL is said to be *Hölder continuous*, if the sections $x \to T_xL$ of the Grassmannian bundles over L are Hölder continuous.

For a subset T_xL and a vector $v \in TM$, let $\operatorname{dist}(v, T_xL) = \min_{u\in T_xL} \|v-u\|$, which means $\operatorname{dist}(v, T_xL)$ is the length of the distance between v and its orthogonal projection of T_xL.

Taken $x, y \in L$, for subbundles T_xL and T_yL, define

$$\operatorname{dist}(T_xL, T_yL) = \max\left\{ \max_{v\in T_xL, \|v\|=1} \operatorname{dist}(v, T_yL), \max_{w\in T_yL, \|w\|=1} \operatorname{dist}(w, T_xL) \right\}.$$

Definition 2.3. For constants $C > 0$ and $\zeta \in (0, 1]$, the tangent bundle TL is said to be $(C, \zeta)-$Hölder continuous, if

$$\operatorname{dist}(T_xL, T_yL) \leq C \operatorname{dist}_L(x, y)^\zeta \quad \text{for all } y \in B(x, \varepsilon) \cap L \textit{ and } x \in U.$$

Here $\operatorname{dist}_L(x, y)$ is the length of geodesic along L connecting x and y. Given a C^1 submanifold $L \subset U$, we define

$$\kappa(L) = \inf\{C > 0 : TL \text{ is } (C, \zeta)\text{-Hölder}\}.$$

The next result on the Hölder control of the tangent direction is all we need. See its proof in [1, Corollary 2.4].

Proposition 2.4. *Given $C_1 > 0$ such that for any C^1 submanifold $L \subset U$ tangent to C^{cu}_a,*

(1) *there is $n_0 \geq 1$, then $\kappa(f^n(L)) \leq C_1$ for every $n \geq n_0$ and $f^k(L) \subset U$ for all $0 \leq k \leq n$;*
(2) *if $\kappa(L) \leq C_1$, then $\kappa(f^n(L)) \leq C_1$ for all $n \geq 1$ and $f^k(L) \subset U$ for all $0 \leq k \leq n$;*
(3) *if L, n are as in the previous item, then the Jacobian functions*

$$J_k : f^k(L) \ni x \mapsto \log|\det\big(Df \mid T_xf^k(L)\big)|, \quad 0 \leq k \leq n,$$

are (J, ζ)-Hölder continuous with $J > 0$ depending only on C_1 and f.

This proposition would be useful in proving Item (3) of Lemma 2.9, i.e the bounded distortion estimates at hyperbolic times in next subsection.

We can derive uniform expansion and bounded distortion from NUE assumption in the centre-unstable direction, with the definition below. Here we do not need the full strength of partially hyperbolic, we only consider the cu-direction has condition (NUE).

Definition 2.5. Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for $x \in K$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \qquad \text{for all } 1 \leq k \leq n.$$

For $n \geq 1$ and $\sigma < 1$, we define

$$H_n(\sigma) = \{x \in K\colon\ n \text{ is a } \sigma\text{-hyperbolic time for } x\ \}.$$

Remark 2.6. Given $\sigma < 1$ and $x \in H_n(\sigma)$, we obtain

$$\|Df^{-k} \mid E^{cu}_{f^n(x)}\| \leq \prod_{j=n-k+1}^{n} \|Df^{-1} \mid E^{cu}_{f^j(x)}\| \leq \sigma^k, \tag{4}$$

which means $Df^{-k} \mid E^{cu}_{f^n(x)}$ is a contraction for $1 \leq k \leq n$.

The next result states the existence of σ-hyperbolic times for points satisfying Definition 2.5 and gives indeed the positive frequency for such points. Its proof can be found in [1, Lemma 3.1, Corollary 3.2].

Proposition 2.7. *There exist $\theta > 0$ and $\sigma > 0$ such that for every $x \in K$ with $\mathcal{E}(x) \leq n$ there exist σ-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq n$ for x with $l \geq \theta n$.*

In the sequel, we consider a fixed σ and simply write H_n instead of $H_n(\sigma)$.

Remark 2.8. If $a > 0$ and $\delta_1 > 0$ are sufficiently small such that the δ_1-neighborhood of K is contained in U, we get by continuity

$$\|Df^{-1}(f(y))v\| \leq \frac{1}{\sqrt{\sigma}}\|Df^{-1}|E^{cu}_{f(x)}\|\,\|v\|, \tag{5}$$

whence $x \in K$, $\operatorname{dist}(y,x) \leq \delta_1$, and $v \in C^{cu}_a(y)$.

For a given disk $\Delta \subset M$, we denote the distance between $x, y \in \Delta$ by $\operatorname{dist}_\Delta(x,y)$, measured along Δ. Let $0 < \delta < \delta_1$ and $n_0 \geq 1$.

Items (1)-(3) in the next result have been proved in [1, Lemma 5.2 & Corollary 5.3], and item (4) is a consequence of item (2).

Lemma 2.9. *Let $\Delta \subset U$ be a C^1 disk of radius δ tangent to the centre-unstable cone field with $\kappa(\Delta) \leq C_1$ and $x \in \Delta \cap K$. There exists $C_2 > 1$ such that if $n \geq n_0$ and $x \in H_n$, then there exists a neighborhood $V_n(x)$ of x and $V_n(x) \subset \Delta$ so that:*

(1) *f^n maps $V_n(x)$ diffeomorphically onto a centre-unstable ball $B(f^n(x), \delta_1)$;*
(2) *for every $1 \leq k \leq n$ and $y, z \in V_n(x)$,*

$$\operatorname{dist}_{f^{n-k}(V_n(x))}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \operatorname{dist}_{f^n(V_n(x))}(f^n(y), f^n(z));$$

(3) *for all $y, z \in V_n(x)$*

$$\log \frac{|\det Df^n \mid T_y\Delta|}{|\det Df^n \mid T_z\Delta|} \leq C_2 \operatorname{dist}_{f^n(D)}(f^n(y), f^n(z))^\zeta;$$

(4) *$V_n(x) \subset B(x, \delta_1\sigma^n)$.*

The sets $V_n(x)$ will be called *hyperbolic pre-balls*, and their images $B(f^n(x), \delta_1)$ called *hyperbolic balls*. Item (3) gives the bounded distortion at hyperbolic times.

3. The GMY structure

In this section we prove the existence of the product structure in the attractor. We essentially describe the geometrical and dynamical nature. This process has three steps. Firstly we prove the existence of a centre-unstable disk Δ whose subsets return to a neighborhood of Δ under forward iteration and the image projects along stable leaves covering Δ completely. Secondly, we define a partition on Δ by these subsets. This construction is inspired essentially by [7, Section 3] and [2, Section 3 & 4]. That is, we generalize the structure in [2] for NUE systems to the partially hyperbolic attractor setting as in [7]. We improve the product structure in [7]; see more specifically Subsection 3.5. Finally we show that the set with a product structure satisfies Definition 1.5.

3.1. **The reference disk.** Let D be a local unstable disk as in the assumption of Theorem A. Given δ_1 as in Remark 2.8, take $0 < \delta_s < \delta_1/2$ such that points in K have local stable manifolds of radius δ_s. In particular, these local stable leaves are contained in U; recall (5).

Definition 3.1. Given a disk $\Delta \subset D$, we define the *cylinder* over Δ

$$\mathcal{C}(\Delta) = \bigcup_{x \in \Delta} W^s_{\delta_s}(x).$$

and consider π be the projection from $\mathcal{C}(\Delta)$ onto Δ along local stable leaves. We say that a center-unstable disk γ^u *u-crosses* $\mathcal{C}(\Delta)$ if $\pi(\gamma^u \cap \mathcal{C}(\Delta)) = \Delta$.

From Lemma 2.9 we know that if $\Delta \subset U$ is a small C^1 disk tangent to the centre-unstable cone field with $\kappa(\Delta) \le C_1$ and $x \in \Delta \cap K$, then for each $x \in H_n$, there is a hyperbolic pre-ball which is sent by f^n diffeomorphically onto the ball $B(f^n(x), \delta_1)$. For technical reasons (see Lemma 3.9) we shall take $\delta_1' \ll \delta_1$ and consider $V_n'(x)$ the part of $V_n(x)$ which is sent by f^n onto $B(f^n(x), \delta_1')$. The sets $V_n'(x)$ will also be called hyperbolic pre-balls.

The next lemma follows from [7, Lemma 3.1 & 3.2].

Lemma 3.2. *There are $\delta_0 > 0$, a point $p \in D$ and $N_0 \ge 1$ such that for each hyperbolic pre-ball $V_n'(x)$ there is $1 \le m \le N_0$ for which $f^{n+m}(V_n'(x))$ u-crosses $\mathcal{C}(\Delta_0)$, where $\Delta_0 = B(p, \delta_0) \subset D$.*

From here on we fix the two center-unstable disks centered at p

$$\Delta_0^0 = \Delta_0 = B(p, \delta_0) \quad \text{and} \quad \Delta_0^1 = B(p, 2\delta_0),$$

and the corresponding cylinders

$$\mathcal{C}_0^i = \bigcup_{x \in \Delta_0^i} W^s_{\delta_s}(x), \qquad \text{for } i = 0, 1. \tag{6}$$

The projections along stable leaves will both be denoted by π.

Remark 3.3. We assume that each disk γ^u u-crossing $\mathcal{C}_0^i$ $(i = 0, 1)$ is a disk centered at a point of $W^s_{\delta_s}(p)$ and with the same radius of Δ_0^i. We ignore the difference of radius caused by the height of the cylinder and the angles of the two dominated splitting bundles. Let the top and bottom components of $\partial \mathcal{C}_0^1$ be denotes by $\partial^u \mathcal{C}_0^1$, i.e. the set of points $z \in \partial \mathcal{C}_0^1$ such that $z \in \partial W^s_{\delta_s}(x)$ for some $x \in \Delta_0$. By the domination property, we may take $\delta_0 > 0$ small enough so that any centre-unstable disk γ^u contained in $\mathcal{C}_0^1$ and intersecting $W^s_{\delta_s/2}(p)$ does not reach $\partial^u \mathcal{C}_0^1$.

Given a hyperbolic pre-ball $V_n'(x)$ and m as in the conclusion of Lemma 3.2 above we define

$$\omega_{n,m}^{i,x} = (f|_{V_n'(x)}^{n+m})^{-1}(f^{n+m}(\Delta_0^i) \cap \mathcal{C}_0^i), \quad i = 0, 1. \tag{7}$$

The sets of the type $\omega_{n,m}^{0,x}$, with $x \in H_n \cap \Delta_0$, are the natural candidates to be in the partition $\mathcal{P}$. In the sequel, sometimes we omit m, i and x in the notation $\omega_{n,m}^{i,x}$ and simply use ω_n to denote some element at step n.

For $k \ge n$, set the *annulus* around the element $\omega_n = \omega_{n,m}^{0,x}$

$$A_k(\omega_n) = \{y \in V_n(x) : 0 \le \operatorname{dist}_D(f^{R(\omega_n)}(y), \Delta_0) \le \delta_0 \sigma^{\frac{k-n}{2}}\}. \tag{8}$$

Obviously

$$A_n(\omega_n) \cup \omega_n = \omega_{n,m}^{1,x}.$$

3.2. **Partition on the reference disk.** In this subsection we describe an algorithm of a (Leb_D mod 0) partition $\mathcal{P}$ of Δ_0. The algorithm is similar to the one in [2], but in the present context of a diffeomorphism, each element of the partition will return to another u-leaf which *u-crosses* $\mathcal{C}_0^0$. Along the process we shall introduce sequences of objects (Δ_n), (Ω_n), (S_n) and (A_n). For each n, Δ_n is the set of points left in the reference disk up to time n and Ω_n is the union of elements of the partition at step n. The set S_n (*satellite*) contains the components which could have been chosen for the partition but are too close to already chosen elements. More precise notation will be shown along the constructing process.

3.2.1. *First step of induction.* Given $n_0 \in \mathbb{N}$ and consider the dynamics after time n_0. Remember $\Delta_0^c = D \setminus \Delta_0$. By the third assertion of [2, Lemma 3.7], there is a finite set of points $I_{n_0} = \{z_1, \ldots, z_{N_{n_0}}\} \in H_{n_0} \cap \Delta_0$ such that

$$H_{n_0} \cap \Delta_0 \subset V'_{n_0}(z_1) \cup \cdots \cup V'_{n_0}(z_{N_{n_0}}).$$

Consider a maximal family of pairwise disjoint sets of type (7) contained in Δ_0,

$$\{\omega_{n_0,m_0}^{1,x_0}, \omega_{n_0,m_1}^{1,x_1}, \ldots, \omega_{n_0,m_{k_{n_0}}}^{1,x_{k_{n_0}}}\},$$

and denote

$$\Omega_{n_0} = \{\omega_{n_0,m_0}^{0,x_0}, \omega_{n_0,m_1}^{0,x_1}, \ldots, \omega_{n_0,m_{k_{n_0}}}^{0,x_{k_{n_0}}}\}$$

These are the elements of the partition $\mathcal{P}$ constructed in the n_0-step of the algorithm. The *recurrence time* $R(\omega_{n_0,m_i}^{0,x_i}) = n_0 + m_i$ with $0 \le i \le k_{n_0}$. Recalling (8), we define

$$A_{n_0}(\Omega_{n_0}) = \bigcup_{\omega \in \Omega_{n_0}} A_{n_0}(\omega).$$

We need to keep track of the sets $\{\omega_{n_0,m}^{1,z} : z \in I_{n_0}, 0 \le m \le N_0\}$ which overlap $\Omega_{n_0} \cup A_{n_0}(\Omega_{n_0})$ or Δ_0^c. Given $\omega \in \Omega_{n_0}$, for each $0 \le m \le N_0$, we define

$$I_{n_0}^m(\omega) = \left\{x \in I_{n_0} : \omega_{n_0,m}^{1,x} \cap (\omega \cup A_{n_0}(\omega)) \neq \emptyset\right\},$$

and the n_0-*satellite* around ω

$$S_{n_0}(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x \in I_{n_0}^m(\omega)} V'_{n_0}(x) \cap (\Delta_0 \setminus \omega), \tag{9}$$

We write

$$S_{n_0}(\Delta_0) = \bigcup_{\omega \in \Omega_{n_0}} S_{n_0}(\omega).$$

Similarly, we define the n_0-satellite associated to $\Delta_0^c = D \setminus \Delta_0$

$$S_{n_0}(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\omega_{n_0,m}^{1,x} \cap \Delta_0^c \neq \emptyset} V'_{n_0}(x) \cap \Delta_0.$$

We will show in the general step, the volume of $S_{n_0}(\Delta_0^c)$ is exponentially small. The 'global' n_0-satellite is

$$S_{n_0} = \bigcup_{\omega\in\Omega_{n_0}} S_{n_0}(\omega) \cup S_{n_0}(\Delta_0^c).$$

The remaining points at step n_0 are

$$\Delta_{n_0} = \Delta_0 \setminus \Omega_{n_0}.$$

Clearly,

$$H_{n_0} \cap \Delta_0 \subset S_{n_0} \cup \Omega_{n_0}.$$

3.2.2. *General step of induction.* The general step of the construction follows the ideas above with minor modifications. As before, there is a finite set of points $I_n = \{z_1, \dots, z_{N_n}\} \in H_n \cap \Delta_0$ such that

$$H_n \cap \Delta_0 \subset V_n'(z_1) \cup \cdots \cup V_n'(z_{N_n}).$$

Assume that the sets Ω_i, Δ_i and S_i are defined for each $i \le n-1$. Assuming

$$\Omega_\ell = \{\omega_{\ell,m_0}^{0,x_0}, \omega_{\ell,m_1}^{0,x_1}, \dots, \omega_{\ell,m_{k_\ell}}^{0,x_{k_\ell}}\}$$

for $n_0 \le \ell \le n-1$, let

$$A_n(\Omega_\ell) = \cup_{\omega\in\Omega_\ell} A_n(\omega).$$

Now we consider a maximal family of pairwise disjoint sets of type (7) contained in Δ_{n-1},

$$\{\omega_{n,m_0}^{1,x_0}, \omega_{n,m_1}^{1,x_1}, \dots, \omega_{n,m_{k_n}}^{1,x_{k_n}}\}$$

satisfying

$$\omega_{n,m}^{1,x_i} \cap \left(\cup_{\ell=n_0}^{n-1}\{A_n(\Omega_\ell) \cup \Omega_\ell\}\right) = \emptyset, \quad i = 1, \dots, k_n,$$

and define

$$\Omega_n = \{\omega_{n,m_0}^{0,x_0}, \omega_{n,m_1}^{0,x_1}, \dots, \omega_{n,m_{k_n}}^{0,x_{k_n}}\}.$$

These are the elements of the partition $\mathcal{P}$ constructed in the n-step of the algorithm. Set $R(x) = n + m_i$ for each $x \in \omega_{n,m_i}^{0,x_i}$ with $0 \le i \le \ell_n$. Given $\omega \in \Omega_{n_0} \cup \cdots \cup \Omega_n$ and $0 \le m \le N_0$, let

$$I_n^m(\omega) = \left\{x \in I_n : \omega_{n,m}^{1,x} \cap (\omega \cup A_n(\omega)) \ne \emptyset\right\},$$

define

$$S_n(\omega) = \bigcup_{m=0}^{N_0} \bigcup_{x\in I_n^m(\omega)} V_n'(x) \cap (\Delta_0 \setminus \omega) \tag{10}$$

and

$$S_n(\Delta_0) = \bigcup_{\omega\in\Omega_{n_0}\cup\cdots\cup\Omega_n} S_n(\omega).$$

Similarly, the *n-satellite* associated to Δ_0^c is

$$S_n(\Delta_0^c) = \bigcup_{m=0}^{N_0} \bigcup_{\omega_{n,m}^{1,x}\cap\Delta_0^c\ne\emptyset} V_n'(x) \cap \Delta_0.$$

Remark 3.4. Observe that the volume of $S_n(\Delta_0^c)$ decays exponentially. Actually, it follows from the definition of $S_n(\Delta_0^c)$ and Lemma 2.9 that

$$S_n(\Delta_0^c) \subset \{x \in \Delta_0 : \mathrm{dist}_D(x, \partial\Delta_0) < 2\delta_0\sigma^{n/2}\}.$$

Thus, we have $\eta > 0$ such that $\mathrm{Leb}_D(S_n(\Delta_0^c)) \leq \eta\sigma^{n/2}$.

Finally we define the *n-satellite* associate to $\Omega_{n_0} \cup \cdots \cup \Omega_n$

$$S_n = S_n(\Delta_0) \cup S_n(\Delta_0^c)$$

and

$$\Delta_n = \Delta_0 \setminus \bigcup_{i=n_0}^{n} \Omega_i.$$

We clearly have

$$H_n \cap \Delta_0 \subset S_n \cup \bigcup_{i=n_0}^{n} \Omega_i. \tag{11}$$

3.3. **Estimates on the satellites.** For the sake of notational simplicity, we shall avoid the superscript 0 in the sets $\omega_{n,m}^{0,x}$. The next lemma shows that, given n and m, the conditional volume of the union of $\omega_{n,m}^{x}$ which intersects one chosen element is proportional to the conditional volume of this element. The proportion constant is uniformly summable with respect to n.

Though we consider here the case of partially hyperbolic attractor, and also the construction is modified a bit, the proofs of the next two lemmas are still essentially the same of [2, Lemmas 4.4 & 4.5].

Lemma 3.5. (1) *There exists $C_3 > 0$ such that, for any $n \geq n_0$, $0 \leq m \leq N_0$, and finitely many points $\{x_1, \ldots, x_N\} \in I_n$ satisfying $\omega_{n,m}^{x_i} = \omega_{n,m}^{x_1}$ $(1 \leq i \leq N)$, we have*

$$\mathrm{Leb}_D\left(\bigcup_{i=1}^{N} V_n'(x_i)\right) \leq C_3 \,\mathrm{Leb}_D(\omega_{n,m}^{x_1}).$$

(2) *There exists $C_4 > 0$ such that for $k \geq n_0$, $\omega \in \Omega_k$ and $0 \leq m \leq N_0$, given any $n \geq k$ we obtain*

$$\mathrm{Leb}_D\left(\bigcup_{x \in I_n^m(\omega)} \omega_{n,m}^{x}\right) \leq C_4\sigma^{\frac{n-k}{2}}\,\mathrm{Leb}(\omega).$$

Proposition 3.6. *There exists $C_5 > 0$, s.t. $\forall\, \omega \in \Omega_k$, and $n \geq k$, we have*

$$\mathrm{Leb}_D(S_n(\omega)) < C_5\sigma^{\frac{n-k}{2}}\,\mathrm{Leb}_D(\omega).$$

Proof. Consider now $k \geq n_0$ and $n \geq k$. Fix $\omega \in \Omega_k$ and consider $S_n(\omega)$ the n-satellite associated to it. By definition of $S_n(\omega)$ and Lemma 3.5 Item (1) we have

$$\mathrm{Leb}_D(S_n(\omega)) \;\leq\; \sum_{m=0}^{N_0} \sum_{x \in I_n^m(\omega)} \mathrm{Leb}_D\left(V_n'(x) \cap (\Delta_0 \setminus \omega)\right) + \mathrm{Leb}_D(V_k'(\omega) \setminus \omega)$$

$$\leq \quad C_3 \sum_{m=0}^{N_0} \mathrm{Leb}_D \left(\bigcup_{x \in I_n^m(\omega)} \omega_{n,m}^x \right) + C_3 \,\mathrm{Leb}_D(\omega).$$

In this last step we have used the obvious fact that for fixed n, m the sets of the form $\omega_{n,m}^x$ with $x \in I_n^m(\omega)$ are pairwise disjoint. Thus, by Lemma 3.5 Item (2),

$$\mathrm{Leb}_D(S_n(\omega)) \leq C_3(C_4(N_0+1)+1)\sigma^{\frac{n-k}{2}} \,\mathrm{Leb}_D(\omega).$$

Let $C_5 = C_3(C_4(N_0+1)+1)$. □

Definition 3.7. Given $k \geq n_0$ and $\omega_{k,m}^x \in \Omega_k$, for some $x \in \Delta_0$ and $0 \leq m \leq N_0$, we define for $n \geq k$

$$B_n^k(x) = S_n(\omega_{k,m}^x) \cup \omega_{k,m}^x \quad \text{and} \quad t(B_n^k(x)) = k.$$

Notice that k and n are both hyperbolic times for points in Δ_0. The set $\omega_{k,m}^x$ will be called the *core* of $B_n^k(x)$ and denoted as $C(B_n^k(x))$.

The next result follows immediately from Proposition 3.6.

Corollary 3.8. *For all $n \geq k$ and x, we have*

$$\mathrm{Leb}_D(B_n^k(x)) \leq (C_5+1)\,\mathrm{Leb}_D(C(B_n^k(x))).$$

The dependence of δ_1' on δ_1 becomes clear in the next lemma.

Lemma 3.9. *If $n_0 \leq k \leq k'$, $n \geq k$, $n' \geq k'$ and $B_n^k(x) \cap B_{n'}^{k'}(y) \neq \emptyset$, then*

$$C(B_n^k(x)) \cup C(B_{n'}^{k'}(y)) \subset V_k(x).$$

Proof. Since k and $n \geq k$ are hyperbolic times, by the second assertion of Lemma 2.9

$$\mathrm{diam}_{f^k(D)}(f^k(B_n^k(x))) \leq 2\delta_1' + 4\delta_1'\sigma^{\frac{n-k}{2}}.$$

Using again the second assertion of Lemma 2.9 we finally have

$$\mathrm{diam}_D(B_n^k(x)) \leq (2\delta_1' + 4\delta_1'\sigma^{\frac{n-k}{2}})\sigma^{\frac{k}{2}} \leq 6\delta_1'\sigma^{\frac{k}{2}}.$$

Similarly

$$\mathrm{diam}_D(B_{n'}^{k'}(y)) \leq 6\delta_1'\sigma^{\frac{k'}{2}}.$$

Now observe that is enough to obtain the conclusion of the lemma for $n = k$ and $n' = k'$. By the computation above we have $\mathrm{diam}_D(B_n^n(x)) \leq 6\delta_1'\sigma^{n/2}$, and $\mathrm{diam}_D(B_{n'}^{n'}(y)) \leq 6\delta_1'\sigma^{n'/2} \leq 6\delta_1'\sigma^{n/2}$. Then we have

$$\mathrm{dist}_{f^n(D)}(f^n(x), \partial f^n(V_n'(y))) \leq 7\delta_1' \ll \delta_1,$$

so $f^n(V_n'(y)) \subset B(f^n(x), \delta_1)$. We build a set $W_n'(y) = f^{-n}(f^n(V_n'(y))) \cap V_n(x)$. By the definition of V_n, f^n is an isomorphism between $W_n'(y)$ and $f^n(V_n'(y))$. But also f^n is an isomorphism between $V_n'(y)$ and $f^n(V_n'(y))$. By the uniqueness in Lemma 2.9, $V_n'(y) = W_n'(y)$. In particular, $V_n'(y) \subset V_n(x)$. And so $C(B_{n'}^{k'}(y)) \subset V_n(x)$. Then

$$C(B_n^k(x)) \cup C(B_{n'}^{k'}(y)) \subset V_n(x).$$

□

Lemma 3.10. *There exists $P \geq N_0$ such that for all $n_0 \leq t_1 \leq t_2$, $B_{t_2+P}^{t_2}(y) \cap B_{t_2+P}^{t_1}(x) = \emptyset$.*

Proof. Suppose, on the contrary, that for all $P \geq N_0$ we have $B^{t_2}_{t_2+P}(y) \cap B^{t_1}_{t_2+P}(x) \neq \emptyset$. Take a point z in the intersection. Then, letting $R_1 = R(C(B^{t_1}_{t_2+P}(x)))$, recalling $t_2 + P$ is a hyperbolic time in the definitions of $B^{t_1}_{t_2+P}(x)$ and $B^{t_2}_{t_2+P}(y)$, by the second assertion of Lemma 2.9 we obtain

$$\mathrm{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B^{t_1}_{t_2+P}(x)))\right) \leq 2\delta_1' \sigma^{\frac{t_2+P-R_1}{2}};$$

and also

$$\mathrm{dist}_{f^{R_1}(D)}\left(f^{R_1}(z), f^{R_1}(C(B^{t_2}_{t_2+P}(y)))\right) \leq 2\delta_1' \sigma^{\frac{t_2+P-R_1}{2}}.$$

Hence,

$$\mathrm{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B^{t_1}_{t_2+P}(x))), f^{R_1}(C(B^{t_2}_{t_2+P}(y)))\right) \leq 4\delta_1' \sigma^{\frac{t_2+P-R_1}{2}}.$$

Letting P large enough such that $4\delta_1'\sigma^{P/2} < \delta_0\sigma^{N_0/2}$, we have

$$\mathrm{dist}_{f^{R_1}(D)}\left(f^{R_1}(C(B^{t_1}_{t_2+P}(x))), f^{R_1}(C(B^{t_2}_{t_2+P}(y)))\right) \leq \delta_0 \sigma^{\frac{t_2-t_1}{2}},$$

which means $C(B^{t_2}_{t_2+P}(y)) \subset A_{t_2}(C(B^{t_1}_{t_1+P}(x)))$. This gives a contradiction. □

3.4. **Tail of recurrence times.** Though our constructions are different from [14], our approach in the estimates below is inspired in [14, Section 3.2]. Given a local unstable disk $D \subset K$ and constants $0 < \tau \leq 1$, $0 < c < 1$, we assume $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. Observe that there exists a constant $\eta > 0$ such that for all $n \in \mathbb{N}$

$$\mathrm{Leb}_D\{x \in D \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq \sigma^{n/2}\} \leq \eta\sigma^{n/2}\,\mathrm{Leb}_D(\Delta_0). \tag{12}$$

Recall Δ_n is the complement at time n, and that θ is defined in Proposition 2.7.

We will show $\mathrm{Leb}_D(\Delta_n)$ decays (stretched) exponentially. That is enough to conclude the proof since $\mathrm{Leb}_D(\{\mathcal{E} > n\})$ is (stretched) exponentially small and $\mathrm{Leb}_D(\{x \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq \sigma^{\frac{\theta n}{2}}\})$ decays exponentially as we know.

Take x in Δ_n, which does not belong either to $\{\mathcal{E} > n\} \cap D$ or to $\{x \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq \sigma^{\frac{\theta n}{2}}\}$. By Proposition 2.7, for n large, x has at least θn hyperbolic times between 1 and n, then at least $\frac{\theta n}{2}$ between $\frac{\theta n}{2}$ and n. We will denote them by $t_1 < \cdots < t_k \leq n$. As now $\mathrm{dist}_D(x, \partial\Delta_0) \geq \sigma^{\frac{\theta n}{2}}$, we have $x \in H'_{t_i}(\Delta_0) := H_{t_i} \cap \Delta_0 \cap \{y \mid \mathrm{dist}(y, \partial\Delta_0) \geq \sigma^{n/2}\}$. If $x \in H'_n(\Delta_0)$, we have $V'_n(x) \subset \Delta_0$ by the second assertion of Lemma 2.9. So we have $x \in S_{t_i}(\Delta_0)$, $i = 1, \ldots, k$. We know from the construction (see (11) in Section 3.2):

$$H_n \cap \Delta_0 \subset S_n \cup \bigcup_{i=n_0}^{n} \Omega_i$$

If $x \notin S_{t_i}$, then $x \notin \Delta_{t_i}$ which means $x \notin \Delta_n$, a contradiction. So $x \in S_{t_i}$. As $x \in H'_{t_i}(\Delta_0)$, $x \in S_{t_i}(\Delta_0)$, $i = 1, \ldots, k$. We obtain $k = \frac{\theta n}{2}$. Thus, x belongs to the set

$$Z\left(\frac{\theta n}{2}, n\right) := \left\{x \mid \exists t_1 < \ldots < t_{\frac{\theta n}{2}} \leq n, x \in \bigcap_{i=1}^{\frac{\theta n}{2}} S_{t_i}(\Delta_0)\right\} \cap \Delta_n.$$

So we have

$$\Delta_n \subset \{x \in \Delta_0 \mid \mathcal{E} > n\} \cup \{x \in \Delta_0 \mid \mathrm{dist}_D(x, \partial\Delta_0) \leq \sigma^{\frac{\theta n}{2}}\} \cup Z(\theta n/2, n).$$

Since the second set has exponentially small measure, by (12), it remains to see that the measure of $Z(\theta n/2, n)$ decays exponentially fast. This follows from Proposition 3.11 below. Observe that if there exists $c' > 0$ such that

$$\mathrm{Leb}_D(\Delta_n) \le \mathcal{O}(e^{-c' n^\gamma}),$$

then, for any large integer k, we have $\mathcal{R}_k = \{R > k\} \subset \Delta_{k-N_0}$, and so

$$\mathrm{Leb}_D(R > k) \le \mathrm{Leb}_D(\Delta_{k-N_0}) = \mathcal{O}(e^{-c'(k-N_0)^\gamma}) = \mathcal{O}(e^{-c' k^\gamma}).$$

The next proposition shows that the set of points contained in finite satellite sets and have not been chosen yet has a measure which decays exponentially.

Proposition 3.11. *Set for integers k, N*

$$Z(k,N) \;=\; \left\{ x \mid \exists t_1 < \ldots < t_k \le N, x \in \bigcap_{i=1}^{k} S_{t_i}(\Delta_0) \cap \Delta_N \right\}.$$

There exist $D_5 > 0$ and $\lambda_5 < 1$ such that, for all N and $1 \le k \le N$,

$$\mathrm{Leb}_D(Z(k,N)) \le D_5 \lambda_5^k \,\mathrm{Leb}_D(\Delta_0).$$

For the proof of this result we need several lemmas that we prove in the sequel. We fix some integer $P' \ge P$ (see P in Lemma 3.10) whose value will be made precise in the proof of Proposition 3.11. In Lemmas 3.12, 3.13 and 3.14 we simply denote $B_i = B^{t_i}_{t_i+m_i}(x)$ for some t_i, x, and $m_i \le P'$.

Lemma 3.12. *Set $E \in \mathbf{N}$. Set*

$$\begin{aligned} Z_1(k,B_0) \;=\; & \Big\{ x \mid \exists B_1', B_1, \ldots, B_r', B_r, \text{so that } \forall 1 \le i \le r,\ t_{i-1} \le t_i' \le t_i - E, B_i \not\subseteq B_i', \\ & \sum_{i=1}^{r} \left[\frac{t_i - t_i'}{E} \right] \ge k \text{ and } x \in \bigcap_{i=0}^{r} B_i \cap \bigcap_{i=1}^{r} B_i' \Big\}. \end{aligned}$$

There is D_1 (independent of E, P'), for all k and B_0

$$\mathrm{Leb}_D(Z_1(k,B_0)) \le D_1 (D_1 \sigma^{E/2})^k \,\mathrm{Leb}_D(C(B_0)).$$

Proof. Choose $D_1 > 0$ large enough such that

$$\frac{1}{1 - D_1^{-1}} (\eta^2 C_2^3) \le D_1 \quad \text{and} \quad C_5 + 1 \le D_1.$$

We will prove the assertion by induction on $k \ge 0$. Take $k = 0$. Recall Corollary 3.8. We obtain

$$\mathrm{Leb}_D(Z_1(0,B_0)) \le \mathrm{Leb}_D(B_0) \le (C_5 + 1)\,\mathrm{Leb}_D(C(B_0)) \le D_1 \,\mathrm{Leb}_D(C(B_0)). \tag{13}$$

Then now $k \ge 1$. By decomposition, we have

$$Z_1(k,B_0) \subset \bigcup_{t=1}^{k} \bigcup_{B_1' \cap B_0 \ne \emptyset} \bigcup_{\substack{B_1' \cap B_1 \ne \emptyset, B_1 \not\subseteq B_1', \\ \left[\frac{t_1 - t_1'}{E}\right] \ge t}} Z_1(k-t, B_1).$$

Let $n = t_1 - t_1'$. Fix some B_1' and then take one from all the possible B_1's. We still call it B_1. It is contained in a ring of size $\sigma^{\frac{n}{2}}$ around B_1'. More precisely, setting $p = t_1'$ and defining $Q_1' = f^p(B_1')$, $Q_1 = f^p(C(B_1'))$, we will show that

$$f^p(B_1) \subset \mathcal{C} := \{y \mid \operatorname{dist}_{f^p(D)}(y, \partial Q_1') \le 6\delta_1'\sigma^{\frac{n}{2}}\}. \tag{14}$$

Since B_1 contains a point of $\partial B_1'$, $f^p(B_1)$ contains a point of $\partial Q_1'$. We obtain

$$\operatorname{diam}_{f^p(D)} f^p(B_1) \le \sigma^{\frac{n}{2}} \operatorname{diam}_{f^{p+n}(D)} f^{p+n}(B_1) \le 6\delta_1'\sigma^{\frac{n}{2}}.$$

Then we get (14). By (12), there is η,

$$\operatorname{Leb}_{f^p(D)} \mathcal{C} \le \eta\sigma^{\frac{n}{2}} \operatorname{Leb}_{f^p(D)}(Q_1').$$

Hence

$$\operatorname{Leb}_{f^p(D)}(f^p(B_1)) \le \eta\sigma^{\frac{n}{2}} \operatorname{Leb}_{f^p(D)}(Q_1').$$

Since $C_5 + 1 \le D_1$,

$$\operatorname{Leb}_D(B_1) \le D_1 \operatorname{Leb}_D(C(B_1)).$$

By the bounded distortion constant C_2, we have

$$\operatorname{Leb}_{f^p(D)}(Q_1') \le C_2 D_1 \operatorname{Leb}_{f^p(D)}(Q_1).$$

Then obviously,

$$\operatorname{Leb}_{f^p(D)}(f^p(B_1)) \le C_2 D_1 \eta\sigma^{\frac{n}{2}} \operatorname{Leb}_{f^p(D)}(Q_1). \tag{15}$$

The cores $C(B_1)$ of those possible B_1's are pairwise disjoint by construction. And importantly, the possible cores $C(B_1)$ must be all contained in $V_p(x_1')$, whence $C(B_1') = \omega_{t_1'}^{x_1'}$ by Lemma 3.9. We know that f^p is a diffeomorphism on $V_p(x_1')$. So $f^p(C(B_1)) \subset f^p(V_p(x_1'))$. As

$$f^p(C(B_1)) \subset f^p(B_1) \subset \mathcal{C},$$

then

$$\sum_{B_1' \cap B_1 \ne \emptyset, \left[\frac{t_1 - t_1'}{E}\right] \ge t} \operatorname{Leb}_{f^p(D)}(f^p(C(B_1))) \le \operatorname{Leb}_{f^p(D)} \mathcal{C}.$$

Remember that $Q_1 = f^p(C(B_1'))$. By (15) we get

$$\sum_{B_1' \cap B_1 \ne \emptyset, \left[\frac{t_1 - t_1'}{E}\right] \ge t} \operatorname{Leb}_{f^p(D)}(f^p(C(B_1))) \;\; \le \;\; C_2 D_1 \eta\sigma^{\frac{n}{2}} \operatorname{Leb}_{f^p(D)}(Q_1).$$

Now, using the bounded distortion constant C_2, we obtain

$$\sum_{B_1' \cap B_1 \ne \emptyset, \left[\frac{t_1 - t_1'}{E}\right] \ge t} \operatorname{Leb}_D(C(B_1)) \;\; \le \;\; D_1 C_2^2 \eta\sigma^{\frac{Et}{2}} \operatorname{Leb}_D(C(B_1')). \tag{16}$$

After that, write $q = t_0$ and $C(B_0) = \omega_{q,m}^x$. The possible sets $C(B_1')$'s are pairwise disjoint by construction, and included in $V_q(x)$ by Lemma 3.9. Indeed, f^q is a diffeomorphism on $V_q(x)$ and its distortion is bounded by C_2. Let $Q_0 := f^q(C(B_0))$, and

$f^q(V_q(x)) = B(f^q(x),\delta_1)$, suppose that $\mathrm{Leb}_{f^q(D)}(B(f^q(x),\delta_1)) \le \eta\,\mathrm{Leb}_{f^q(D)}(f^q(C(B_0)))$. By bounded distortion, obtaining by

$$\mathrm{Leb}_D(V_q(x)) \le C_2\,\mathrm{Leb}_D(C(B_0))\frac{\mathrm{Leb}_{f^q(D)}(B(f^q(x),\delta_1))}{\mathrm{Leb}_{f^q(D)}(f^q(C(B_0)))},$$

we have

$$\sum_{B_1'\cap B_0\neq\emptyset} \mathrm{Leb}_D(C(B_1')) \le \eta C_2\,\mathrm{Leb}_D(C(B_0)).$$

Finally, the induction assumption gives

$$\begin{aligned}
\mathrm{Leb}_D(Z_1(k,B_0)) &\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset}\sum_{\left[\frac{t_1-t_1'}{E}\right]\ge t} \mathrm{Leb}_D(Z_1(k-t,B_1))\\
&\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset}\sum_{\left[\frac{t_1-t_1'}{E}\right]\ge t} D_1(D_1\sigma^{E/2})^{k-t}\,\mathrm{Leb}_D(C(B_1))\\
&\le \sum_{t=1}^{k}\sum_{B_1'\cap B_0\neq\emptyset} D_1(D_1\sigma^{\frac{E}{2}})^{k-t}D_1C_2^2\eta\sigma^{\frac{Nt}{2}}\,\mathrm{Leb}_D(C(B_1'))\\
&\le D_1(D_1\sigma^{\frac{E}{2}})^{k}D_1C_2^3\eta^2\sum_{t=1}^{k}(D_1)^{-t}\,\mathrm{Leb}_D(C(B_0)).
\end{aligned}$$

By the definition of D_1, we have $D_1\eta^2C_2^3(\sum_{t=1}^{k}(D_1)^{-t}) \le 1$. Then we get

$$\mathrm{Leb}_D(Z_1(k,B_0)) \le D_1(D_1\sigma^{\frac{E}{2}})^{k}\,\mathrm{Leb}_D(C(B_0)),$$

which ends the proof. □

Lemma 3.13. *Set*

$$Z_2(k,N) = \{x \mid \exists B_1 \supsetneq B_2 \ldots \supsetneq B_k \text{ with } t_1 < \cdots < t_k \le N \text{ and } x \in B_1\cap\ldots\cap B_k\cap\Delta_N\}.$$

Then there exists $\lambda_2 < 1$ such that for all $N \ge 1$ and $1 \le k \le N$,

$$\mathrm{Leb}_D(Z_2(k,N)) \le \lambda_2^k\,\mathrm{Leb}_D(\Delta_0).$$

Proof. We assume N is fixed in this proof, so $Z_2(k) := Z_2(k,N)$. We will prove that the conclusion of the lemma holds for $\lambda_2 = \frac{D_1}{D_1+1}$. By Corollary 3.8 and $C_5+1 \le D_1$, for each possible B, we get

$$\mathrm{Leb}_D(B) \quad\le\quad D_1\,\mathrm{Leb}_D(C(B)). \tag{17}$$

We define $\mathcal{Q}_1$ as a maximal class of sets B with $t(B) \le N$ and not contained in any other B's. Consider $\mathcal{Q}_2 \subset \mathcal{Q}_1^c$ as the class of sets B with $t(B) \le N$ which are included in elements of $\mathcal{Q}_1$. Next we define $\mathcal{Q}_3 \subset \mathcal{Q}_2^c$ as the class of sets B with $t(B) \le N$ which are included in elements of $\mathcal{Q}_2$. We proceed inductively. Notice that this process must stop in a finite number of steps because we always take $t(B) \le N$. We say that an element in $\mathcal{Q}_i$ has *rank* i.

Let now

$$G_k = \bigcup_{i=1}^{k} \bigcup_{B \in \mathcal{Q}_k} C(B),$$

and

$$\tilde{Z}_2(k) = \left(\bigcup_{B \in \mathcal{Q}_k} B \right) \setminus G_k.$$

Now we prove that $Z_2(k) \subset \tilde{Z}_2(k)$. Given $x \in Z_2(k)$, we have $x \in B_1 \cap \ldots \cap B_k \cap \Delta_N$ with $B_1 \supsetneq B_2 \ldots \supsetneq B_k$ and $t(B_k) \leq N$. We clearly have that B_k is of rank $r \geq k$. Take $B'_1 \supsetneq B'_2 \ldots \supsetneq B'_{r-1} \supsetneq B'_r$ a sequence with $B'_i \in \mathcal{Q}_i$ and $B'_r = B_k$. In particular, $x \in B'_i$ for $i = 1, \ldots, k$, and so $x \in \bigcup_{B \in \mathcal{Q}_k} B$. On the other hand, since $x \in \Delta_N$ and $G_k \cap \Delta_N = \emptyset$, we get $x \notin G_k$. So $x \in \tilde{Z}_2(k)$.

Now we deduce the relation between $\mathrm{Leb}_D(\tilde{Z}_2(k+1))$ and $\mathrm{Leb}_D(\tilde{Z}_2(k))$, in such a way that we may estimate $\mathrm{Leb}_D(\tilde{Z}_2(k))$. Take $B \in \mathcal{Q}_{k+1}$. Let B' be an element of rank k containing B. As the cores are pairwise disjoint by nature, $C(B) \cap G_k = \emptyset$. We obtain $C(B) \subset B' \setminus G_k \subset \tilde{Z}_2(k)$. By definition $C(B) \subset G_{k+1}$, whence $C(B) \cap \tilde{Z}_2(k+1) = \emptyset$. This means that $C(B) \subset \tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1)$. Finally, by (17),

$$\begin{aligned}
\mathrm{Leb}_D(\tilde{Z}_2(k+1)) &\leq \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(B) \\
&\leq D_1 \sum_{B \in \mathcal{Q}_{k+1}} \mathrm{Leb}_D(C(B)) \\
&\leq D_1 \, \mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1))
\end{aligned}$$

since the $C(B)$ are pairwise disjoint. Then, we obtain

$$\begin{aligned}
(D_1 + 1) \, \mathrm{Leb}_D(\tilde{Z}_2(k+1)) &\leq D_1 \, \mathrm{Leb}_D(\tilde{Z}_2(k+1)) + D_1 \, \mathrm{Leb}_D(\tilde{Z}_2(k) \setminus \tilde{Z}_2(k+1)) \\
&= D_1 \, \mathrm{Leb}_D(\tilde{Z}_2(k)).
\end{aligned}$$

It yields $\mathrm{Leb}_D(\tilde{Z}_2(k)) \leq \left(\frac{D_1}{D_1+1} \right)^k \mathrm{Leb}_D(\Delta_0)$ by induction. Since $Z_2(k) \subset \tilde{Z}_2(k)$, the same inequality holds for $Z_2(k)$. This concludes the proof. $\square$

The results of Lemma 3.12 and Lemma 3.13 are enough for us to assert next lemma:

Lemma 3.14. *Set*

$$Z_3(k, N) = \left\{ x \mid \exists t_1 < \ldots < t_k \leq N, x \in S_{t_1+m_1}(\Omega_{t_1}) \cap \ldots \cap S_{t_k+m_k}(\Omega_{t_k}) \cap \Delta_N \right\},$$

whence $m_1, \ldots, m_q < P'$. There are constants $D_3 > 0$ and $\lambda_3 < 1$ (both independent of P') such that, for all N and $1 \leq k \leq N$,

$$\mathrm{Leb}_D(Z_3(k, N)) \leq D_3 \lambda_3^k \, \mathrm{Leb}_D(\Delta_0).$$

Proof. Choose E large enough s.t. $D_1 \sigma^{E/2} < 1$ (recall Lemma 3.12). Let us write $N = rE + s$ with $s < E$. Given an arbitrary $x \in Z_3(k, N)$, then there exist instants $t_1 < \ldots < t_k$ as in the definition of $Z_3(k, N)$. For $0 \leq u < r$, take from each interval $[uE, (u+1)E)$ the first appeared $t_i \in \{t_1, \ldots, t_k\}$ (if there is at least one). Denote the got subsequence t_i's by

$t_{1'} < \cdots < t_{k'}$. Since $t_1 < \cdots < t_k \le N$, we can see $k' \ge [\frac{k}{E}]$, which means $Ek' + E \ge k$. Keeping only the instants with odd indexes, we get a sequence of instants $u_1 < \ldots < u_\ell$ with $2\ell \ge k'$, and necessarily $\ell \ge \frac{k-E}{2E}$. Moreover, we have $u_{i+1} - u_i \ge E$ for $1 \le i \le \ell$ by construction.

Now, according to our construction process, we know that associated to each instant u_i there must be some set B_i such that $x \in B_i$, for $1 \le i \le \ell$. Define

$$I = \{1 \le i \le \ell, B_i \subset B_1 \cap \cdots \cap B_{i-1}\} \quad \text{and} \quad J = [1, \ell] \setminus I.$$

If $\#I \ge \ell/2$, we keep only the elements with indexes in I. Recalling Z_2 in Lemma 3.13, we have $x \in Z_2(\ell/2, N)$. Then $Z_2(\ell/2, N)$ has an exponentially small measure in ℓ (then in k). Otherwise, if $\#I \le \ell/2$, then $\#J \ge \ell/2$. Let $j_0 = \sup J$ and $i_0 = \inf\{i < j_0, B_{j_0} \not\subseteq B_i\}$. Let $j_1 = \sup\{j \le i_0, j \in J\}$, $i_1 = \inf\{i < j_1, B_{j_1} \not\subseteq B_i\}$, and continue the process. The process must necessarily stop at some step i_n. Then $J \subset \cup_{s=0}^n (i_s, j_s]$ by construction. We obtain $\sum_{s=0}^n (j_s - i_s) \ge \#J \ge \ell/2$, which shows that

$$\sum_{s=0}^{n} \left[\frac{t(B_{j_s}) - t(B_{i_s})}{E}\right] = \sum_{s=0}^{n} \left[\frac{u_{j_s} - u_{i_s}}{E}\right] \ge \ell/2,$$

since $|u_j - u_i| \ge E(j-i)$ by the process. Hence $x \in Z_1(\ell/2, B_{i_n})$ with the sequence $B_{i_n}, B_{i_n}, B_{j_n}, \ldots, B_{i_0}, B_{j_0}$. As the cores are pairwise disjoint by nature, we use the estimate of Lemma 3.12 and, summing over all the possible B'_{i_n}s, we get

$$\mathrm{Leb}_D(Z_3(k, N)) \le D_3 \lambda_3^k \, \mathrm{Leb}_D(\Delta_0).$$

□

Lemma 3.15. *Given $B_1 = B_{t_1}^{t_1}(x_1)$, let*

$$Z_4(n_1, \ldots, n_k, B_1) = \Big\{ x \mid \exists\, t_2, \ldots, t_k \text{ with } t_1 < \ldots < t_k \text{ and } x_2, \ldots, x_k, \\ \text{s.t. } x \in \bigcap_{i=1}^{k} B_{t_i+n_i}^{t_i}(x_i) \cap \Delta_N \Big\}.$$

Then, there is $D_4 > 0$ (independent of $B_1, n_1, \ldots, n_k$) such that for $n_1, \ldots, n_k > P$,

$$\mathrm{Leb}_D(Z_4(n_1, \ldots, n_k, B_1)) \le D_4(D_4 \sigma^{n_1/2}) \ldots (D_4 \sigma^{n_k/2}) \, \mathrm{Leb}_D(C(B_1)).$$

Proof. The proof is by induction on k. Taking $D_4 > {C_5}^{1/2}$ (recall C_5 in Proposition 3.6), we get the result immediately when $k = 1$. Now suppose $k > 1$. Let $x \in Z_4(n_1, \ldots, n_k, B_1)$. There exists $B_2 = B_{t_2}^{t_2}(x_2)$ constructed at an instant $t_2 > t_1$, and $x \in Z_4(n_2, \ldots, n_k, B_2)$. Suppose

$$\mathrm{Leb}_D(Z_4(n_2, \ldots, n_k, B_2)) \le D_4(D_4 \sigma^{n_2/2}) \ldots (D_4 \sigma^{n_k/2}) \, \mathrm{Leb}_D(C(B_2)).$$

There exists P given by Lemma 3.10, such that $B_{t_2+P}^{t_2}(x_2) \cap B_{t_2+P}^{t_1}(x_1) = \emptyset$. But for all $1 \le i \le k$, we have $x \in B_{t_i+n_i}^{t_i}(x_i)$. So, $t_1 + n_1 < t_2 + P$, i.e. $t_2 - t_1 > n_1 - P$. By the uniform expansion at hyperbolic times, we get

$$\mathrm{diam}_{f^{t_1}(D)}(f^{t_1}(B_2)) \le \sigma^{\frac{t_2 - t_1}{2}} \, \mathrm{diam}_{f^{t_2}(D)}(f^{t_2}(B_2)) \le 6\delta_1' \sigma^{\frac{n_1 - P}{2}}.$$

On the other hand, setting $Q = f^{t_1}(C(B_1))$, we have $\mathrm{dist}_{f^{t_1}(D)}(f^{t_1}(x), \partial Q) \le 2\delta_1' \sigma^{\frac{n_1}{2}}$ when $x \in B^{t_1}_{t_1+n_1}(x_1) \cap B_2$. Then, taking $D_4 \ge 2\delta_1' + 6\delta_1'\sigma^{-P}$ we have

$$f^{t_1}(B_2) \subset \mathcal{C} := \{y \mid \mathrm{dist}_{f^{t_1}(D)}(y, \partial Q) \le D_4 \sigma^{\frac{n_1}{2}}\}.$$

By induction and bounded distortion, we get

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_2,\dots,n_k,B_2))) \le C_2 D_4 (D_4\sigma^{n_2/2})\dots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2))).$$

The possible cores $C(B_2)$'s are pairwise disjoint by nature and contained in $V_{t_1}(x_1)$ by Lemma 3.9. The sets $f^{t_1}(C(B_2))$ are still pairwise disjoint, since f^{t_1} is injective on $V_{t_1}(x_1)$. So they are all contained in the annulus $\mathcal{C}$. We have

$$\begin{aligned}
\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_1,\dots,n_k,B_1))) &\le \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_2,\dots,n_k,B_2)))\\
&\le C_2 D_4 (D_4\sigma^{n_2/2})\dots(D_4\sigma^{n_k/2}) \sum_{B_2} \mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(C(B_2)))\\
&\le C_2 D_4 (D_4\sigma^{n_2/2})\dots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}).
\end{aligned}$$

By (14) and (15), we similarly get $\mathrm{Leb}_{f^{t_1}(D)}(\mathcal{C}) \le C_2 D_1 \eta \sigma^{n_1/2}\,\mathrm{Leb}_{f^{t_1}(D)}(Q)$ whence $Q = f^{t_1}(C(B_1))$. Hence,

$$\mathrm{Leb}_{f^{t_1}(D)}(f^{t_1}(Z_4(n_1,\dots,n_k,B_1))) \le C_2^2 D_1 D_4 \eta \sigma^{n_1/2} (D_4\sigma^{n_2/2})\dots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_{f^{t_1}(D)}(Q).$$

By the bounded distortion constant C_2 of the map f^{t_1} on $V_{t_1}(x_1)$, we get

$$\mathrm{Leb}_D(Z_4(n_1,\dots,n_k,B_1)) \le C_2^3 D_1 \eta (D_4\sigma^{n_1/2})(D_4\sigma^{n_2/2})\dots(D_4\sigma^{n_k/2})\,\mathrm{Leb}_D(C(B_1)).$$

Taking $D_4 \ge C_2^3 D_1 \eta$, we conclude the proof. □

Now we are ready to complete the proof of the metric estimates.

Proof of Proposition 3.11. Take $P' \ge P$ (recall P in Lemma 3.10) so that

$$\sigma^{1/2} + D_3 \sigma^{P'/2} < 1.$$

Let $x \in Z(k, N)$, and consider all the instants u_i for which x is in some $S_{u_i+n_i}(\omega^y_{u_i,m})$ with $n_i \ge P'$, ordered so that $u_1 < \dots < u_p$. Then $x \in Z_4(n_1,\dots,n_p,B_1)$ for some B_1. If $\sum_{i=1}^p n_i \ge k/2$, we conclude the proof. Otherwise, $\sum_{i=1}^p n_i < k/2$, and $p < k/2P'$. Let $v_1 < \dots < v_q$ be other instants for which $x \in S_{v_i+m_i}(\omega^z_{v_i,\tilde m})$, for times $m_1,\dots,m_q < P'$. We have $p + q \ge k$, then $q \ge \frac{(2P'-1)k}{2P'} \ge \frac{k}{2P'}$, whence $P' > 1$. This shows $P'q \ge \frac{k}{2}$.

Thus we have

$$Z(k,N) \subset \bigcup_{B_1} \bigcup_{\substack{n_1,\dots,n_p \ge P',\\ \sum n_i \ge \frac{k}{2}}} Z_4(n_1,\dots,n_p,B_1) \cup Z_3\left(\frac{k}{2P'}, N\right).$$

By Lemma 3.14 and 3.15, we obtain

$$\mathrm{Leb}_D(Z(k,N)) \le \sum_{B_1} \sum_{\substack{n_1,\dots,n_p \ge P',\\ \sum n_i \ge \frac{k}{2}}} D_3 (D_3\sigma^{n_1/2})\dots(D_3\sigma^{n_p/2})\,\mathrm{Leb}_D(C(B_1)) + D_3 \lambda_3^{\frac{k}{2P'}}\,\mathrm{Leb}_D(\Delta_0).$$

We know $\sum_{B_1} \mathrm{Leb}_D(C(B_1)) \leq \mathrm{Leb}_D(\Delta_0) < \infty$ because the cores $C(B_1)$ are pairwise disjoint. What is left to show

$$\sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} (D_3\sigma^{n_1/2}) \dots (D_3\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_1))$$

is exponentially small. Let us adopt

$$\sum_n \sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i = n}} (D_3\sigma^{n_1/2}) \dots (D_3\sigma^{n_p/2}) z^n = \sum_{p=1}^{\infty} \left(D_3 \sum_{n=P'}^{\infty} \sigma^{n/2} z^n \right)^p = \frac{D_3\sigma^{P'/2} z^{P'}}{1 - \sigma^{1/2} z - D_3 \sigma^{P'/2} z^{P'}}.$$

Under the hypothesis $\sigma^{1/2} + D_3\sigma^{P'/2} < 1$, the function above has no extreme pole in the unit disk's neighbourhood in $\mathbb{C}$. Thus its coefficients decay exponentially fast. There are constants $D_5 > 0$ and $\lambda_5 < 1$ such that

$$\sum_{\substack{n_1,\dots,n_p \geq P', \\ \sum n_i \geq \frac{k}{2}}} (D_3\sigma^{n_1/2}) \dots (D_3\sigma^{n_p/2}) \,\mathrm{Leb}_D(C(B_1)) \leq D_5 \lambda_5^n \,\mathrm{Leb}_D(\Delta_0).$$

We sum over $n \geq k/2$ to end the proof. □

3.5. **Product structure.** Consider the center-unstable disk $\Delta_0 \subset D$ and the partition $\mathcal{P}$ of Δ_0 (Leb_D mod 0) defined in Section 3.2. We define

$$\Gamma^s = \left\{ W^s_{\delta_s}(x) : x \in \Delta_0 \right\}.$$

And we define the family of unstable leaves Γ^u as the set of all local unstable leaves intersecting $\mathcal{C}^0$ (recall equation (6) in Section 3.2) which u-cross Δ_0. Clearly Γ^u is nonempty because $\Delta_0 \in \Gamma^u$. It is necessary to prove that Γ^u is compact. By the domination property and Ascoli-Arzela Theorem, any limit leaf Δ_∞ of leaves in Γ^u is a u-disk and u-crossing Δ_0, at the same time it is contained in $\mathcal{C}^0$ since $\mathcal{C}^0$ is closed. As the definition of Γ^u, we can see $\Delta_\infty \in \Gamma^u$. So Γ^u is compact.

Relatively, the *s-subsets* are as the following: we define $\mathcal{C}(\omega)$ as the cylinder made by the stable leaves passing through the points in ω, i.e.

$$\mathcal{C}(\omega) = \bigcup_{x \in \omega} W^s_{\delta_s}(x).$$

The pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \dots$ are the sets $\{\mathcal{C}(\omega) \cap \Gamma^u\}_{\omega \in \mathcal{P}}$.

Then we should check that $f^{R_i}(\Lambda_i)$ is *u-subset*. Given an element $\omega \in \mathcal{P}$, by construction there is some $R(\omega) \in \mathbb{N}$ such that $f^{R(\omega)}(\omega)$ is a center-unstable disk u-crossing $\mathcal{C}^0$. Since each γ^u is a copy of Δ_0 but with a different center, and very important that, $\Gamma^u \cap \mathcal{C}(\omega) \in \cup_{x \in \omega} W^s_{\delta_s}(x)$. Since by construction $f^{R(\omega)}(\omega)$ intersects $W^s_{\delta_s/4}(p)$, then according to the choice of δ_0 and the invariance of the stable foliation, we have that each element of $f^{R(\omega)}(\mathcal{C}(\omega) \cap \Gamma^u)$ must u-cross $\mathcal{C}^0$, and is contained in the $\lambda^{R(\omega)}\delta_s$ height neighborhood of $f^{R(\omega)}(\omega)$. Ignore the difference caused by the angle. We can say it is contained in $\mathcal{C}^0$. So, that is a *u-subset.*

In the sequel, the *product structure* $\Lambda = \Gamma^u \cap \Gamma^s$ will be proven as a *GMY structure.* Observe that the set Λ coincides with the union of the leaves in Γ^u. We can diminish it so

that we say $\Lambda \subset K$ as the assertion of Theorem A. Properties $(\mathbf{P_0})$ until $(\mathbf{P_2})$ are satisfied by nature. In the following we prove $(\mathbf{P_3})$. The proof of $(\mathbf{P_4})$ is a repeat of that in [7].

3.6. **Uniform expansion and bounded distortion.** Here we prove property $(\mathbf{P_3})$(a).

Lemma 3.16. *There is $C > 0$ such that, given $\omega \in \mathcal{P}$ and $\gamma \in \Gamma^u$, we have for all $1 \le k \le R(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$*

$$\mathrm{dist}_{f^{R(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)-k}(x), f^{R(\omega)-k}(y)) \le C\sigma^{k/2}\, \mathrm{dist}_{f^{R(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{R(\omega)}(x), f^{R(\omega)}(y)).$$

Proof. Let ω be an element of partition $\mathcal{P}$ constructed in the Section 3.2. So there are a point $x \in D$ with σ-hyperbolic time $n(\omega)$ satisfying $R(\omega) - N_0 \le n(\omega) \le R(\omega)$. Since we take $\delta_s, \delta_0 < \delta_1/2$, it follows from (5) that $n(\omega)$ is a $\sqrt{\sigma}$-hyperbolic time for every point in $\mathcal{C}(\omega) \cap \gamma$. By (4), we obtain that for all $1 \le k \le n(\omega)$ and all $x, y \in \mathcal{C}(\omega) \cap \gamma$

$$\mathrm{dist}_{f^{n(\omega)-k}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)-k}(x), f^{n(\omega)-k}(y)) \le \sigma^{k/2}\, \mathrm{dist}_{f^{n(\omega)}(\mathcal{C}(\omega)\cap\gamma)}(f^{n(\omega)}(x), f^{n(\omega)}(y)).$$

As $R(\omega) - n(\omega) \le N_0$, the result follows with C depending only on N_0 and the the derivative of f. □

Property $(\mathbf{P_3})$(b) follows from Proposition 2.4 together with Lemma 3.16 as in [1, Proposition 2.8]. For the sake of completeness we prove it here.

Lemma 3.17. *There is $\bar{C} > 0$ such that, for all $x, y \in \Lambda_i$ with $y \in \gamma^u(x)$, we have*

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} \le \bar{C}\, \mathrm{dist}(f^{R_i}(x), f^{R_i}(y))^\zeta.$$

Proof. For $0 \le i < R_i$ and $y \in \gamma^u(x) \in \Gamma^u$, we denote $J_k(y) = |\det D(f^k)^u(f^k(y))|$. Then,

$$\log \frac{\det D(f^{R_i})^u(x)}{\det D(f^{R_i})^u(y)} = \sum_{k=0}^{R_i-1} (J_k(x) - J_k(y)) \le \sum_{k=0}^{R_i-1} L_1\, \mathrm{dist}_D(f^k(x), f^k(y))^\zeta.$$

By Lemma 3.16, the sum of all $\mathrm{dist}_D(f^k(x), f^k(y))^\zeta$ over $0 \le k \le R_i$ is bounded by

$$\mathrm{dist}_D(f^{R_i}(x), f^{R_i}(y))^\zeta/(1 - \sigma^{\zeta/2}).$$

It suffices to take $\bar{C} = L_1(1 - \sigma^{\zeta/2})$ □

3.7. **Regularity of the foliations.** $(\mathbf{P_4})$ has been proved in [7]. This is standard for uniformly hyperbolic attractors, and it follows adapting classical ideas to our setting. $(\mathbf{P_4})$(a) flows from the next result whose proof may be found in [7, Corollary 3.8].

Proposition 3.18. *There are $C > 0$ and $0 < \beta < 1$ such that for all $y \in \gamma^s(x)$ and $n \ge 0$*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \le C\beta^n.$$

For $(\mathbf{P_4})$(b), we start by introducing some useful notions. We say that $\phi : N \to G$, where N and G are submanifolds of M, is *absolutely continuous* if it is an injective map for which there exists $J : N \to \mathbb{R}$, called the *Jacobian* of ϕ, such that

$$\mathrm{Leb}_G(\phi(A)) = \int_A J d\, \mathrm{Leb}_N\,.$$

Finally, property ($\mathbf{P_4}$)(b) follows from the next result whose proof may be found in [7, Proposition 3.9].

Proposition 3.19. *Given $\gamma, \gamma' \in \Gamma^u$, define $\phi\colon \gamma' \to \gamma$ by $\phi(x) = \gamma^s(x) \cap \gamma$. Then ϕ is absolutely continuous and the Jacobian of ϕ is given by*

$$J(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\phi(x)))}.$$

We deduce from Proposition 3.18 that this infinite product converges uniformly.

4. Application

Here we present a open robust class of partially hyperbolic diffeomorphisms (or, more generally, diffeomorphisms with a dominated splitting) whose centre-unstable direction is non-uniformly expanding at Lebesgue almost everywhere in M. The example was introduced in [1, Appendix] as the following: assume $K = M$, through deformation of a uniformly hyperbolic map by isotopy inside some small region, we can prove the new map satisfies the condition (NUE) in the cu-direction. Then we prove $\mathrm{Leb}_D\{\mathcal{E} > n\} = \mathcal{O}(e^{-cn^\tau})$. We sketch the main steps.

We consider a linear Anosov diffeomorphism f_0 on the d-dimensional torus $M = T^d$, $d \geq 2$. Thus we have the hyperbolic splitting $TM = E^u \oplus E^s$. Let $V \subset M$ be some small compact domain, such that $f_0|_V$ is injective. Let $\pi : \mathbb{R}^d \to T^d$ be the canonical projection, there exist unit open cubes K^0, K^1 in $\mathbb{R}^d$ such that $V \subset \pi(K^0)$ and $f_0(V) \subset \pi(K^1)$. We obtain f in a sufficiently small C^1-neighborhood of f_0, and f satisfies the assumptions of Theorem A, such that:

(1) f admits invariant cone field C^{cu} and C^s, with small width $\alpha > 0$ and containing, respectively, the unstable bundle E^u and E^s of the Anosov diffeomorphism f_0;
(2) f is *volume expanding everywhere*: there is $\sigma_1 > 0$ such that $|\det(Df|T_x\mathcal{D}^{cu})| > \sigma_1$ for any $x \in M$ and any disk $\mathcal{D}^{cu}$ through x tangent to the center-unstable cone field C^{cu};
(3) f is C^1-close to f_0 in the compliment of V, so that f^{cu} is *expanding outside* V: there is $\sigma_2 < 1$ satisfying $\|(Df|T_x\mathcal{D}^{cu})^{-1}\| < \sigma_2$ for $x \in M \setminus V$ and any disks $\mathcal{D}^{cu}$ tangent to C^{cu};
(4) f^{cu} is *not too contracting* on V: there is small $\delta_0 > 0$ satisfying $\|(Df|T_x\mathcal{D}^{cu})^{-1}\| < 1 + \delta_0$ for any $x \in V$ and any disks $\mathcal{D}^{cu}$ tangent to C^{cu}.

Let $\mathcal{F}_0^u$ be the unstable foliation of f_0, and $\mathcal{F}_j = f^j(\mathcal{F}_0^u)$ for all $j \geq 0$. By Item (1), each $\mathcal{F}_j$ is a foliation of T^d tangent to the centre-unstable cone field C^{cu}. For any subset E of a leaf of $\mathcal{F}_j$, $j \geq 0$, we denote $\mathrm{Leb}_j(E)$ the Lebesgue measure of E inside the leaf. Let us fix any small disk D_0 contained in a leaf of $\mathcal{F}_0$. See the proof of the following lemma in [1, Appendix] Lemma A.1.

Lemma 4.1. *Let $B_1, \ldots, B_p, B_{p+1} = V$ be an arbitrary partition of M such that f is injective on B_j, for $1 \leq j \leq p+1$. There exist $\theta > 0$ such that, the orbit of Lebesgue almost every $x \in D_0$ spends a fraction θ of the time in $B_1 \cup \ldots \cup B_p$:*

$$\#\{0 \leq j < n : f^j(x) \in B_1 \cup \ldots \cup B_p\} \geq \theta n$$

for every large n.

By Lemma 4.1, we conclude that Leb_{D_0}-almost every point $x \in D_0$ spends a positive fraction θ of the time outside the domain V. Then by Item (3) and (4) above, there exists $c_0 > 0$,

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \le -c_0$$

for Leb_{D_0}-almost every point $x \in D_0$. since D_0 was an arbitrary disk intersect foliations $\mathcal{F}_0^s$ transversely, and $\mathcal{F}_0^s$ is an absolutely continuous foliation, we say f is non-uniformly expanding along E^{cu}, at Lebesgue almost everywhere in $M = T^d$. Moreover, the induced Lebesgue measure of the set

$$\{x \in D_0 : \|Df^j(x)^{-1}\| > e^{-c_0 j} \text{ for some } j \ge n\}$$

is exponentially small. Then there exists a constant $0 < d < 1$, we have

$$\mathrm{Leb}_{D_0}\{\mathcal{E} > n\} = \mathcal{O}(e^{-dn}).$$

References

[1] J. F. Alves, C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, Invent. Math. **140** (2000), 351-398.

[2] J. F. Alves, C. L. Dias, S. Luzzatto, *Geometry of expanding absolutely continuous invariant measures and the liftability problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[3] J. F. Alves, J. M. Freitas, S. Luzzatto, S. Vaienti, *From rates of mixing to recurrence times via large deviations*, Adv. Math. **228**, n.2 (2011), 1203-1236.

[4] J. F. Alves, *SRB measures for non-hyperbolic systems with multidimensional expansion*, Ann. Scient. Éc. Norm. Sup. (4) **33** (2000), 1-32.

[5] J. F. Alves, S. Luzzatto, V. Pinheiro, *Markov structures and decay of correlations for non-uniformly expanding dynamical systems*, Ann. Inst. Henri Poincaré (C) Non Linear Anal. **22**, n.6 (2005) 817-839.

[6] J. F. Alves, V. Pinheiro, *Slow rates of mixing for dynamical systems with hyperbolic structures*, J. Stat. Phys. **131**, n.3 (2008) 505-534.

[7] J. F. Alves, V. Pinheiro, *Gibbs-Markov structures and limit laws for partially hyperbolic attractors with mostly expanding central direction*, Adv. Math. 223 (2010) 1706-1730.

[8] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. (1970) vol. 92 pp. 725-747.

[9] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. Springer Verlag, (1975) vol. 470.

[10] C. Bonatti, M. Viana, *SRB measures for partially hyperbolic systems with mostly contracting central direction*, Israel J. Math. **115** (2000), 157-193.

[11] M. Benedicks, L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque **261** (2000), 13-56.

[12] A. A. Castro, *Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting central direction*, Israel J. of Math. **130** (2001), 29-75.

[13] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, Trans. Amer. Math. Soc. **356**, no. 4 (2004), 1637–1689.

[14] S. Gouëzel, *Decay of correlations for nonuniformly expanding systems*, Bull. Soc. Math. France **134**, n.1 (2006), 1-31.

[15] I. Melbourne, M. Nicol, *Large deviations for nonuniformly hyperbolic systems*, Trans. Amer. Math. Soc. **360** (2008) 6661-6676.

[16] V. Pinheiro, *Sinai-Ruelle-Bowen measures for weakly expanding maps*, Nonlinearity **19** (2006) 1185-1200

[17] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, Inst. Hautes Études Sci. Publ. Math. **50** (1979) 27-58.
[18] Y. Sinai. *Markov partitions and C-diffeomorphisms.* Funkcional. Anal. i Prilo zen (1968) vol. 2 (1) pp. 64-89.
[19] A. Tahzibi, *Stably ergodic diffeomorphisms which are not partially hyperbolic*, Israel J. Math. **142** (2004), 315–344.
[20] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. Math. **147** (1998), 585-650.
[21] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153-188.

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